Deletion of imperfect cloned copies

2004 J. Phys. A: Math. Gen. 37 11877

(http://iopscience.iop.org/0305-4470/37/49/007)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 11877–11887 PII: S0305-4470(04)86930-0

Deletion of imperfect cloned copies

Satyabrata Adhikari and Binayak S Choudhury

Department of Mathematics, Bengal Engineering College (Deemed University), Howrah-711 103, West Bengal, India

E-mail: bsc@math.becs.ac.in and satyyabrata@yahoo.com

Received 9 June 2004
Published 24 November 2004
Online at stacks.iop.org/JPhysA/37/11877
doi:10.1088/0305-4470/37/49/007

Abstract

In this work, we design a deleting machine and show that, for some given condition on machine parameters, it gives a slightly better result than the Pati–Braunstein (PB) deleting machine (2000 *Nature* **404** 164, 2001 *Preprint* quant-ph/0007121). Also it is shown that for some particular values of the machine parameters it acts like the PB deleting machine. We also study the combined effect of a cloning and deleting machine, where first the cloning is done by some standard cloning machines such as Wootters–Zurek (1982 *Nature* **299** 802) and Buzek–Hillery (1996 *Phys. Rev.* A **54** 1844) cloning machines and then the copy mode is deleted by the PB deleting machine or our prescribed deleting machine. We then examine the distortion of the input state and the fidelity of deletion.

PACS numbers: 03.67.−a, 03.65.Bz, 89.70.+c

In quantum information theory, it is well known that an unknown quantum state cannot be cloned or deleted [1, 4, 5]. But we cannot rule out the possibility of constructing an approximate cloning machine [1–3]. We can divide approximate cloning machines into two categories: (i) state-dependent cloning machine: a cloning machine that depends on input state such as the Wootter–Zurek (WZ) cloning machine [1]. It can clone better for some states while it gives a worse clone for some other states. (ii) Universal quantum cloning machine: a cloning machine which does not depend on the input state, such as the Buzek–Hillery (BH) cloning machine [2]. The fidelity of cloning is same for all input states while cloning with this cloning machine.

Like cloning machines, deleting machines have not performed well. This was first observed by Pati and Braunstein and they showed that the linearity of quantum theory does not allow one to delete a copy of an arbitrary quantum state perfectly. But ignoring the problem of perfect deletion we can construct approximate deleting machines which are input state dependent, such as the Pati–Braunstein deleting machine. These deleting machines are not perfect in the sense that they can neither delete the copy mode perfectly nor retain the input state. It may happen that we first clone an unknown quantum state by using a known cloning machine and then after using the copy mode, we want to delete it with a known deleting machine. After completing the whole procedure, it is natural to ask about the fidelity of deletion and the distortion of the input state. We try to give the answer to the above question in this paper.

In section 1, we briefly discuss the Wootter–Zurek quantum copying machine and the Buzek–Hillery universal quantum cloning machine. In section 2, we briefly discuss the Pati–Braunstein deleting machine and D Qiu's non-optimal universal quantum deleting machine [8]. In section 3, we construct a quantum deleting machine which is input state dependent. Then we show that the minimum average distortion of the input qubit and maximum fidelity of deletion approach $\frac{1}{3}$ and $\frac{5}{6}$, respectively. In section 4, we study the concatenation of cloning and deleting machines. Lastly, we prescribe the transformation rule of the general deleting machine.

Section 1

Wootter–Zurek (WZ) copying machine

The Wootters and Zurek (WZ) quantum copying machine defined by the transformation relation on the basis vector $|0\rangle$ and $|1\rangle$ is given by

$$|0\rangle|Q\rangle \rightarrow |0\rangle|0\rangle|Q_0\rangle \tag{1.1a}$$

$$|1\rangle|Q\rangle \rightarrow |1\rangle|1\rangle|Q_1\rangle. \tag{1.1b}$$

Unitarity of the transformation gives

$$\langle Q|Q\rangle = \langle Q_0|Q_0\rangle = \langle Q_1|Q_1\rangle = 1. \tag{1.2}$$

Let an unknown quantum state be given by

$$|s\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{1.3}$$

Without any loss of generality, we may assume α and β are real numbers and $\alpha^2 + \beta^2 = 1$.

The density matrix of $|s\rangle$ is

$$\begin{aligned}\rho^{\text{id}} &= |s\rangle\langle s| \\ &= \alpha^2|0\rangle\langle 0| + \alpha\beta|0\rangle\langle 1| + \alpha\beta|1\rangle\langle 0| + \beta^2|1\rangle\langle 1|.\end{aligned} \tag{1.4}$$

Using the transformation relation (1.1), we obtain

$$|s\rangle|Q\rangle \rightarrow \alpha|0\rangle|0\rangle|Q_0\rangle + \beta|1\rangle|1\rangle|Q_1\rangle \equiv |\psi\rangle^{(\text{out})}. \tag{1.5}$$

If it is assumed that the two copying machine states $|Q_0\rangle$ and $|Q_1\rangle$ are orthonormal i.e., $\langle Q_0|Q_1\rangle = 0$.

Then the reduced density operator $\rho_{\text{ab}}{}^{\text{out}}$ is given by

$$\rho_{\text{ab}}{}^{\text{out}} = \text{Tr}_{\text{x}}\left[\rho_{\text{abx}}{}^{\text{out}}\right] = \text{Tr}_{\text{x}}[|\psi\rangle^{(\text{out})\,(\text{out})}\langle\psi|] = \alpha^2|00\rangle\langle 00| + \beta^2|11\rangle\langle 11|. \tag{1.6}$$

The density operators of the final state in the original mode and the copy mode are given by

$$\rho_{\text{a}}{}^{(\text{out})} = \text{Tr}_{\text{b}}\left[\rho_{\text{ab}}{}^{\text{out}}\right] = \alpha^2|0\rangle\langle 0| + \beta^2|1\rangle\langle 1| \tag{1.7a}$$

$$\rho_{\text{b}}{}^{(\text{out})} = \text{Tr}_{\text{a}}\left[\rho_{\text{ab}}{}^{\text{out}}\right] = \alpha^2|0\rangle\langle 0| + \beta^2|1\rangle\langle 1|. \tag{1.7b}$$

The copying quality, i.e., the distance between the density matrix of the input state $\rho_{\text{a}}{}^{\text{id}}$ and the output state of the original mode $\rho_{\text{a}}{}^{\text{out}}$ can be measured by the Hilbert–Schmidt norm.

The Hilbert–Schmidt norm is defined as

$$D_{\text{a}} \equiv \text{Tr}\left[\rho_{\text{a}}{}^{\text{id}} - \rho_{\text{a}}{}^{(\text{out})}\right]^2. \tag{1.8}$$

There are also other measures like Bures metric and trace norm [7]. But comparatively, the Hilbert–Schmidt norm is easier to calculate and it also serves as a good measure of quantifying the distance between the pure states.

Therefore the Hilbert–Schmidt norm for the density operators given by equations (1.4) and (1.7) is

$$D_1 = 2\alpha^2\beta^2 = 2\alpha^2(1-\alpha^2). \tag{1.9}$$

Since D_1 depends on α^2, so the WZ cloning machine is state dependent. For some values of α it copies well while for some states it operates badly.

Buzek–Hillery (BH) copying machine

In Buzek and Hillery (BH) cloning, the transformation rule [2] is given by

$$|0\rangle|Q\rangle \rightarrow |0\rangle|0\rangle|Q_0\rangle + [|0\rangle|1\rangle + |1\rangle|0\rangle]|Y_0\rangle \tag{1.10a}$$

$$|1\rangle|Q\rangle \rightarrow |1\rangle|1\rangle|Q_1\rangle + [|0\rangle|1\rangle + |1\rangle|0\rangle]|Y_1\rangle. \tag{1.10b}$$

Unitarity of the transformation gives

$$\begin{aligned}&\langle Q_i|Q_i\rangle + 2\langle Y_i|Y_i\rangle = 1, \qquad i = 0, 1\\ &\langle Y_0|Y_1\rangle = \langle Y_1|Y_0\rangle = 0.\end{aligned} \tag{1.11}$$

If further it is assumed that

$$\begin{aligned}&\langle Q_i|Y_i\rangle = 0, \qquad i = 0, 1\\ &\langle Q_0|Q_1\rangle = 0\end{aligned} \tag{1.12}$$

then the density operator of the output state after copying procedure is

$$\begin{aligned}\rho_{ab}{}^{out} = {}& \alpha^2|00\rangle\langle 00|\langle Q_0|Q_0\rangle + \sqrt{2}\alpha\beta|00\rangle\langle +|\langle Y_1|Q_0\rangle + \sqrt{2}\alpha\beta|+\rangle\langle 00|\langle Q_0|Y_1\rangle\\ &+ [2\alpha^2\langle Y_0|Y_0\rangle + 2\beta^2\langle Y_1|Y_1\rangle]|+\rangle\langle +| + \sqrt{2}\alpha\beta|+\rangle\langle 11|\langle Q_1|Y_0\rangle\\ &+ \sqrt{2}\alpha\beta|11\rangle\langle +|\langle Y_0|Q_1\rangle + \beta^2|11\rangle\langle 11|\langle Q_1|Q_1\rangle\end{aligned} \tag{1.13}$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$.

The density operator describing the original mode can be obtained by taking the partial trace over the copy mode and it reads

$$\rho_a{}^{(out)} = |0\rangle\langle 0|[\alpha^2 + \xi(\beta^2 - \alpha^2)] + |0\rangle\langle 1|\alpha\beta\eta + |1\rangle\langle 0|\alpha\beta\eta + |1\rangle\langle 1|[\beta^2 + \xi(\alpha^2 - \beta^2)] \tag{1.14}$$

where $\langle Y_0|Y_0\rangle = \langle Y_1|Y_1\rangle \equiv \xi$

$$\langle Y_0|Q_1\rangle = \langle Q_0|Y_1\rangle = \langle Q_1|Y_0\rangle = \langle Y_1|Q_0\rangle \equiv \frac{\eta}{2}. \tag{1.15}$$

The density operator $\rho_b{}^{(out)}$ describes that the copy mode is exactly the same as the density operator describes the original mode $\rho_a{}^{(out)}$.

Now the Hilbert–Schmidt norm for the density operators (1.4) and (1.14) is given by

$$D_2 = 2\xi^2(4\alpha^4 - 4\alpha^2 + 1) + 2\alpha^2\beta^2(\eta - 1)^2. \tag{1.16}$$

It is found that D_2 is input state independent if ξ and η are related by

$$\eta = 1 - 2\xi. \tag{1.17}$$

Therefore,

$$D_2 = 2\xi^2. \tag{1.18}$$

If $\frac{\partial D_{ab}^{(2)}}{\partial \alpha^2} = 0$, then the cloning machine is input state independent for $\xi = \frac{1}{6}$, where

$$D_{ab}^{(2)} = \mathrm{Tr}\big[\rho_{ab}{}^{(out)} - \rho_{ab}{}^{(id)}\big]^2 \qquad \text{and} \qquad \rho_{ab}{}^{id} = \rho_a{}^{id} \otimes \rho_b{}^{id}. \tag{1.19}$$

Section 2

Pati and Braunstein [6] defined a deleting transformation for the orthogonal qubit:

$$|0\rangle|0\rangle|A\rangle \rightarrow |0\rangle|\Sigma\rangle|A_0\rangle \tag{2.1a}$$

$$|1\rangle|1\rangle|A\rangle \rightarrow |1\rangle|\Sigma\rangle|A_1\rangle \tag{2.1b}$$

$$|0\rangle|1\rangle|A\rangle \rightarrow |0\rangle|1\rangle|A\rangle \tag{2.1c}$$

$$|1\rangle|0\rangle|A\rangle \rightarrow |1\rangle|0\rangle|A\rangle \tag{2.1d}$$

where $|\Sigma\rangle$ represents some standard state, $|A\rangle$ is the initial state and $|A_0\rangle$, $|A_1\rangle$ are the final states of the ancilla.

After operating deleting machine (2.1) on the input state $|s\rangle|s\rangle$ where $|s\rangle$ (1.3), the reduced density matrix obtained by taking partial trace over the machine mode 'c' is given by

$$\begin{aligned}\rho_{ab} &= \mathrm{tr}_c(|s\rangle|s\rangle\langle s|\langle s|)\\ &= |\alpha|^4|0\rangle\langle 0| \otimes |\Sigma\rangle\langle\Sigma| + |\beta|^4|1\rangle\langle 1| \otimes |\Sigma\rangle\langle\Sigma| + 2|\alpha|^2|\beta|^2|\psi^+\rangle\langle\psi^+|\end{aligned} \tag{2.2}$$

where $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |01\rangle)$.

The reduced density matrix for the qubit in the mode b will be

$$\rho_b = \mathrm{tr}_a(\rho_{ab}) = (1 - 2|\alpha|^2|\beta|^2)|\Sigma\rangle\langle\Sigma| + |\alpha|^2|\beta|^2 I \tag{2.3}$$

where $I = |0\rangle\langle 0| + |1\rangle\langle 1|$.

The fidelity of deletion is found to be

$$F_b = \langle\Sigma|\rho_b|\Sigma\rangle = (1 - |\alpha|^2|\beta|^2). \tag{2.4}$$

Since F_b depends on $|\alpha|^2$, the average fidelity of deletion is given by $F_b = \int F_b\,\mathrm{d}\alpha^2 = \frac{5}{6}$.

The reduced density matrix for the qubit in the mode a will be

$$\rho_a = \mathrm{tr}_b(\rho_{ab}) = |\alpha|^4|0\rangle\langle 0| + |\beta|^4|1\rangle\langle 1| + |\alpha|^2|\beta|^2 I. \tag{2.5}$$

The fidelity of the qubit in mode a is

$$F_a = \langle\psi|\rho_a|\psi\rangle = 1 - 2|\alpha|^2|\beta|^2. \tag{2.6}$$

The average fidelity in this case is $\frac{2}{3}$.

The PB deleting machine is a state-dependent deleting machine, since it depends on the input state. Also it is found that the average fidelity for the first qubit in mode 'a' is less than the actual deleting mode 'b'. This shows that linearity of quantum theory not only prohibits the deletion of an unknown state but also restricts the other qubit to retain its original state. The authors also proved that unitarity does not allow one to delete copies of two non-orthogonal states exactly.

Recently, D Qiu gave a transformation rule [8] for the universal quantum deleting machine, which is given below

$$U|0\rangle|0\rangle|Q\rangle \rightarrow |0\rangle|\Sigma\rangle|A_0\rangle + |1\rangle|0\rangle|B_0\rangle \tag{2.7a}$$

$$U|1\rangle|1\rangle|Q\rangle \rightarrow |1\rangle|\Sigma\rangle|A_1\rangle + |0\rangle|1\rangle|B_1\rangle \tag{2.7b}$$

$$U|0\rangle|1\rangle|Q\rangle \rightarrow |0\rangle|1\rangle|C_0\rangle \tag{2.7c}$$

$$U|1\rangle|0\rangle|Q\rangle \rightarrow |1\rangle|0\rangle|C_1\rangle. \tag{2.7d}$$

Based on some assumptions and calculations, he verified that such a universal quantum deleting machine does not exist.

Then he constructed a deleting machine which works as a universal quantum deleting machine given by

$$U|0\rangle|0\rangle|Q\rangle \rightarrow a_0|0\rangle|A_0\rangle + b_0|1\rangle|B_0\rangle \tag{2.8a}$$

$$U|1\rangle|1\rangle|Q\rangle \to a_1|1\rangle|A_1\rangle + b_1|0\rangle|B_1\rangle \tag{2.8b}$$

$$U|0\rangle|1\rangle|Q\rangle \to |0\rangle|1\rangle \tag{2.8c}$$

$$U|1\rangle|0\rangle|Q\rangle \to |1\rangle|0\rangle \tag{2.8d}$$

where $|Q\rangle$ represents the ancilla state and $|A_i\rangle$, $|B_i\rangle$ $(i = 0, 1)$ are the final states of the ancilla.

This deleting machine may play an important role when the memory in a quantum computer is inadequate. He also showed that the above prescribed deleting machine is input state independent or universal in the sense that the distance $D(|\alpha|^2) = \mathrm{tr}\big[\big(\rho_2{}^{(\mathrm{out})} - |\psi\rangle\langle\psi|\big)^t\big(\rho_2{}^{(\mathrm{out})} - |\psi\rangle\langle\psi|\big)\big]$ is input state independent, where $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and

$$\rho_2{}^{(\mathrm{out})} = \mathrm{tr}_1(U|\psi\rangle|\psi\rangle|Q\rangle\langle Q|\langle\psi|\langle\psi|U^t).$$

But the above deleting machine (2.8) is a non-optimal universal quantum deleting machine. The machine is non-optimal in the sense that it gives low fidelity of deletion and it cannot be improved.

Section 3

In this work, we prescribe a deleting machine given by

$$U|0\rangle|0\rangle|Q\rangle \to |0\rangle|\Sigma\rangle|A_0\rangle \tag{3.1a}$$

$$U|0\rangle|1\rangle|Q\rangle \to (a_0|0\rangle|1\rangle + b_0|1\rangle|0\rangle)|Q\rangle \tag{3.1b}$$

$$U|1\rangle|0\rangle|Q\rangle \to (a_1|0\rangle|1\rangle + b_1|1\rangle|0\rangle)|Q\rangle \tag{3.1c}$$

$$U|1\rangle|1\rangle|Q\rangle \to |1\rangle|\Sigma\rangle|A_1\rangle \tag{3.1d}$$

where $|Q\rangle$, $|A_0\rangle$, $|A_1\rangle$ and $|\Sigma\rangle$ have their usual meaning and a_i, b_i $(i = 0, 1)$ are the complex numbers.

Due to the unitarity of the transformation (3.1) the following relations hold:

$$\left.\begin{array}{ll} \langle A_i|A_i\rangle = 1 & (i - 0, 1) \\ |a_i|^2 + |b_i|^2 = 1 & (i = 0, 1) \\ a_i a_{1-i}^* + b_i b_{1-i}^* = 0 & (i = 0, 1) \\ \langle A_1|Q\rangle = \langle A_0|Q\rangle = 0. & \end{array}\right\} \tag{3.2}$$

Further, we assume that

$$\langle A_1|A_0\rangle = \langle A_0|A_1\rangle = 0. \tag{3.3}$$

A general pure state is given by

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad \alpha^2 + \beta^2 = 1 \tag{3.4}$$

where without any loss of generality we can assume that α and β are real numbers.

Using the transformation relation (3.1) and exploiting the linearity of U, we have

$$\begin{aligned} U|\psi\rangle|\psi\rangle|Q\rangle &= \alpha^2 U|0\rangle|0\rangle|Q\rangle + \alpha\beta U|0\rangle|1\rangle|Q\rangle + \alpha\beta U|1\rangle|0\rangle|Q\rangle + \beta^2 U|1\rangle|1\rangle|Q\rangle \\ &= \alpha^2|0\rangle|\Sigma\rangle|A_0\rangle + \alpha\beta[g|0\rangle|1\rangle + h|1\rangle|0\rangle]|Q\rangle + \beta^2|1\rangle|\Sigma\rangle|A_1\rangle \\ &\equiv |\psi\rangle_{12}{}^{(\mathrm{out})} \end{aligned} \tag{3.5}$$

where $g = a_0 + a_1$, $h = b_0 + b_1$.

The reduced density operator of the output state in modes '1' and '2' is given by

$$\rho_1{}^{(\mathrm{out})} = \mathrm{Tr}_2\left[\rho_{12}{}^{(\mathrm{out})}\right] = \mathrm{Tr}_2\left[|\psi\rangle_{12}{}^{(\mathrm{out})\,(\mathrm{out})}{}_{12}\langle\psi|\right]$$
$$= [\alpha^4 + \alpha^2\beta^2 gg^*]|0\rangle\langle 0| + [\beta^4 + \alpha^2\beta^2 hh^*]|1\rangle\langle 1| \tag{3.6a}$$

$$\rho_2{}^{(\mathrm{out})} = \mathrm{Tr}_1\left[\rho_{12}{}^{(\mathrm{out})}\right] = \mathrm{Tr}_1\left[|\psi\rangle_{12}{}^{(\mathrm{out})\,(\mathrm{out})}{}_{12}\langle\psi|\right]$$
$$= \alpha^4|\Sigma\rangle\langle\Sigma| + \alpha^2\beta^2[gg^*|1\rangle\langle 1| + hh^*|0\rangle\langle 0|] + \beta^4|\Sigma\rangle\langle\Sigma|. \tag{3.6b}$$

Now to see the performance of our machine, we must calculate the distortion of the input state and the fidelity of deletion.

Therefore, the distance between the density operators $\rho_{\mathrm{a}}{}^{(\mathrm{id})} = |\psi\rangle\langle\psi|$ and (3.6*a*) is

$$D_1(\alpha^2) = \mathrm{Tr}\left[\rho_1{}^{(\mathrm{out})} - \rho_{\mathrm{a}}{}^{(\mathrm{id})}\right]^2$$
$$= k\alpha^4\beta^4 + 2\alpha^2\beta^2$$

where $k = (gg^* - 1)^2 + (hh^* - 1)^2$.

Since D_1 depends on α^2, so average distortion of input qubit in mode 1 is given by

$$\overline{D}_1 = \int_0^1 D_1(\alpha^2)\,\mathrm{d}\alpha^2 = \frac{1}{3}\left(1 + \frac{(gg^* - 1)^2 + (hh^* - 1)^2}{10}\right). \tag{3.6c}$$

The reduced density matrix of the qubit in mode 2 contains an error due to imperfect deleting and the error can be measured by calculating the fidelity. Thus, the fidelity is given by

$$F_1 = \langle\Sigma|\rho_2|\Sigma\rangle = 1 - k_1\alpha^2\beta^2$$

where $k_1 = 2 - gg^*M_2 - hh^*\,(1 - M^2)$, $M = \langle\Sigma|1\rangle$.

Since fidelity of deletion depends on the input state, so the average fidelity over all input state is given by

$$\overline{F}_1 = \int_0^1 F_1(\alpha^2)\,\mathrm{d}\alpha^2$$
$$= 1 - \frac{k_1}{6} = \frac{2}{3} + \frac{(gg^* - hh^*)M^2 + hh^*}{6}. \tag{3.6d}$$

From equations (3.6*c*) and (3.6*d*), we observe that the minimum average distortion of the state in mode '1' from the input state is $\frac{1}{3}$ and the minimum average fidelity of deletion is $\frac{2}{3}$. So our prime task is to construct a deleting machine or, in other words, to find the value of the machine parameter a_0, a_1, b_0, b_1 which maximize the fidelity of deletion but keep the average distortion at its minimum value.

To solve the above discussed problem, we take $gg^* - hh^* = \varepsilon$ and $hh^* = 1 + \varepsilon_1$, where ε and ε_1 are very small quantities. Then equations (3.6*c*) and (3.6*d*) give

$$\overline{D}_1 = \frac{1}{3} + \frac{(\varepsilon_1)^2 + (\varepsilon + \varepsilon_1)^2}{30} \qquad \overline{F_1} = \frac{5}{6} + \frac{\varepsilon M^2 + \varepsilon_1}{6}.$$

Therefore, $\overline{D}_1 \to \frac{1}{3}$, $\overline{F_1} \to \frac{5}{6}$ as $\varepsilon, \varepsilon_1 \to 0$.

The above equation shows that if we choose machine parameters a_0, a_1, b_0, b_1 in such a way that gg^* and hh^* both are very close to unity then only we are able to keep the distortion at its minimum level and increase the average fidelity to $\frac{5}{6}$.

Section 4

In this section, we study the effect of deleting machines after cloning imperfect copies of an unknown quantum state by a cloning machine such as the WZ cloning machine and the BH deleting machine. The concatenations of cloning and deleting machines are different from identity transformation in the sense that the distortion of one qubit from its original state is not zero and the fidelity of deletion of another qubit is not unity. Otherwise the distortion and the fidelity of deletion is found out to be 0 and 1, respectively. This happens only when the copy is cloned perfectly and from the perfectly cloned copies, if we can delete the copy mode perfectly. But this case cannot arise since linearity of quantum theory prohibits perfect cloning and perfect deletion.

WZ cloning machine and PB deleting machine

Let an unknown quantum state (3.4) be cloned by the WZ cloning machine.

Using cloning transformation (1.1), an unknown quantum state (3.4) cloned to

$$\alpha|0\rangle|0\rangle|Q_0\rangle + \beta|1\rangle|1\rangle|Q_1\rangle. \tag{4.1}$$

Now, operating deleting machine (2.1) to the cloned state (4.1), we get the final output state as

$$|\phi\rangle_{\mathrm{xy}}{}^{(\mathrm{out})} = \alpha|0\rangle|\Sigma\rangle|A_0\rangle + \beta|1\rangle|\Sigma\rangle|A_1\rangle. \tag{4.2}$$

The reduced density operator describing the output state in modes x and y is given by

$$\begin{aligned}\rho_{\mathrm{x}}{}^{(\mathrm{out})} &= \mathrm{Tr}_{\mathrm{y}}(\rho_{\mathrm{xy}}) \\ &= \alpha^2|0\rangle\langle 0| + \beta^2|1\rangle\langle 1| \end{aligned}\tag{4.3a}$$

$$\rho_{\mathrm{y}}{}^{(\mathrm{out})} = \mathrm{Tr}_{\mathrm{x}}(\rho_{\mathrm{xy}}) = |\Sigma\rangle\langle\Sigma|. \tag{4.3b}$$

The distance between the density operators $\rho_{\mathrm{a}}{}^{(\mathrm{id})} = |\psi\rangle\langle\psi|$ and (4.3*a*) is

$$\begin{aligned}D_3(\alpha^2) &= \mathrm{Tr}\big[\rho_{\mathrm{x}}{}^{(\mathrm{out})} - \rho_{\mathrm{a}}{}^{(\mathrm{id})}\big]^2 \\ &= 2\alpha^2(1-\alpha^2). \end{aligned}\tag{4.4}$$

The average distortion of input qubit after cloning and deleting operation is given by

$$\begin{aligned}\overline{D}_3 &= \int_0^1 D_3(\alpha^2)\,\mathrm{d}\alpha^2 \\ &= 0.33. \end{aligned}\tag{4.5}$$

The fidelity of deletion is given by

$$F_3 = \langle\Sigma|\rho_{\mathrm{y}}|\Sigma\rangle = 1. \tag{4.6}$$

The above equations show that if we clone an unknown quantum state by the WZ cloning machine, and delete a copy qubit by the PB deleting machine then the fidelity of deletion is found to be 1 but the concatenation of the cloning and deleting machine cannot retain the input qubit in its original state.

BH cloning machine and PB deleting machine

Let an unknown quantum state (3.4) be cloned by the BH cloning machine.

Using cloning transformation (1.10), quantum state (3.4) cloned to

$$\alpha[|0\rangle|0\rangle|Q_0\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_0\rangle] + \beta[|1\rangle|1\rangle|Q_1\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_1\rangle]. \tag{4.7}$$

After operating deleting machine (2.1) to the cloned state (4.7), the output state is given by

$$|\phi\rangle_{xy}{}^{(out)} = \frac{1}{\sqrt{1+2\xi}}\{\alpha[|0\rangle|\Sigma\rangle|A_0\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_0\rangle] + \beta[|1\rangle|\Sigma\rangle|A_1\rangle + (|0\rangle|1\rangle + |1\rangle|0\rangle)|Y_1\rangle]\}. \tag{4.8}$$

The reduced density operator describing the output state in modes x and y is given by

$$\rho_x{}^{(out)} = \mathrm{Tr}_y(\rho_{xy}) = \mathrm{Tr}_y\big(|\phi\rangle_{xy}{}^{(out)\,(out)}{}_{xy}\langle\phi|\big) = \frac{1}{1+2\xi}\{|0\rangle\langle 0|[\alpha^2+\xi] + |1\rangle\langle 1|[\beta^2+\xi]\} \tag{4.9a}$$

$$\rho_y{}^{(out)} = \mathrm{Tr}_x(\rho_{xy}) = \mathrm{Tr}_x\big(|\phi\rangle_{xy}{}^{(out)\,(out)}\langle\phi|\big) = \frac{1}{1+2\xi}\{|\Sigma\rangle\langle\Sigma| + I\xi\} \tag{4.9b}$$

where I is the identity matrix in two-dimensional Hilbert space.

The distance between the density operators $\rho_a^{(id)} = |\psi\rangle\langle\psi|$ and (4.9*a*) is

$$D_4(\alpha^4) = \mathrm{Tr}\big[\rho_x{}^{(out)} - \rho_a{}^{(id)}\big]^2 = \frac{2\xi^2 + 2\alpha^2\beta^2(1+4\xi)}{(1+2\xi)^2}. \tag{4.10}$$

The average distortion of the input state is given by

$$\overline{D_4} = \int_0^1 D_4(\alpha^2)\,\mathrm{d}\alpha^2 = \frac{6\xi^2+4\xi+1}{3(1+2\xi)^2} = \frac{11}{32},$$

for the BH cloning machine $\xi = \frac{1}{6}$.

Since we are using the BH cloning machine to clone an unknown quantum state, therefore the fidelity of deletion is given by

$$F_4 = \langle\Sigma|\rho_y|\Sigma\rangle = \frac{1+\xi}{1+2\xi} = \frac{7}{8},$$

for the BH cloning machine $\xi = \frac{1}{6}$.

WZ cloning machine and deleting machine (3.1)

After operating deleting machine (3.1) to the cloned state (4.1), we get the output state as

$$|\phi\rangle_{xy}{}^{(out)} = \alpha|0\rangle|\Sigma\rangle|A_0\rangle + \beta|1\rangle|\Sigma\rangle|A_1\rangle \tag{4.11}$$

The reduced density operator describing the output state in modes x and y is given by

$$\rho_x{}^{(out)} = \mathrm{Tr}_y(\rho_{xy}) = \mathrm{Tr}_y\left(|\phi\rangle_{xy}{}^{(out)\,(out)}{}_{xy}\langle\phi|\right) = \alpha^2|0\rangle\langle 0| + \beta^2|1\rangle\langle 1| \tag{4.12a}$$

$$\rho_y{}^{(out)} = \mathrm{Tr}_x(\rho_{xy}) = \mathrm{Tr}_x\left(|\phi\rangle_{xy}{}^{(out)\,(out)}{}_{xy}\langle\phi|\right) = |\Sigma\rangle\langle\Sigma| \tag{4.12b}$$

The distance between the density operators $\rho_a{}^{(id)} = |\psi\rangle\langle\psi|$ and (4.14*a*) is

$$D_5(\alpha^2) = \mathrm{Tr}\big[\rho_x{}^{(out)} - \rho_a{}^{(id)}\big]^2 = 2\alpha^2(1-\alpha^2) \tag{4.13}$$

Since D_5 depends on α^2, so the average distortion of deletion is given by

$$\overline{D_5} = \int_0^1 D_5(\alpha^2)\,\mathrm{d}\alpha^2 = 0.33 \tag{4.14}$$

The fidelity of the second qubit is given by

$$F_5 = \langle\Sigma|\rho_\mathrm{y}|\Sigma\rangle = 1. \tag{4.15}$$

BH cloning machine and deleting machine (3.1)

After operating deleting machine (3.1) to the cloned state (4.7), we get

$$|\phi\rangle_\mathrm{xy}{}^{(\mathrm{out})} = \{\alpha[|0\rangle|\Sigma\rangle|A_0\rangle + (g|0\rangle|1\rangle + h|1\rangle|0\rangle)|Y_0\rangle] + \beta[|1\rangle|\Sigma\rangle|A_1\rangle + (g|0\rangle|1\rangle + h|1\rangle|0\rangle)|Y_1\rangle]\}. \tag{4.16}$$

We assume

$$\langle A_0|Y_0\rangle = \langle A_1|Y_1\rangle = 0. \tag{4.17}$$

The reduced density operators describing the output state in two different modes is given by

$$\rho_\mathrm{x}{}^{(\mathrm{out})} = \mathrm{Tr}_\mathrm{y}(\rho_\mathrm{xy}) = \mathrm{Tr}_\mathrm{y}\big(|\phi\rangle_\mathrm{xy}{}^{(\mathrm{out})\,(\mathrm{out})}{}_\mathrm{xy}\langle\phi|\big) = \frac{1}{[1 + (gg^* + hh^*)\xi]}\{|0\rangle\langle 0|(\alpha^2 + \xi gg^*) + |1\rangle\langle 1|(\beta^2 + \xi hh^*)\} \tag{4.18a}$$

$$\rho_\mathrm{y}{}^{(\mathrm{out})} = \mathrm{Tr}_\mathrm{x}(\rho_\mathrm{xy}) = \mathrm{Tr}_\mathrm{x}\big(|\phi\rangle_\mathrm{xy}{}^{(\mathrm{out})\,(\mathrm{out})}{}_\mathrm{xy}\langle\phi|\big) = \frac{1}{[1 + (gg^* + hh^*)\xi]}\{|\Sigma\rangle\langle\Sigma| + |0\rangle\langle 0|(\xi hh^*) + |1\rangle\langle 1|(\xi gg^*)\}. \tag{4.18b}$$

Now in order to measure the degree of distortion, we evaluate the distance between the density operators (4.18*a*) and (1.4) given by

$$D_6(\alpha^2) = \mathrm{Tr}\big[\rho_\mathrm{x}{}^{(\mathrm{out})} - \rho_\mathrm{a}{}^{(\mathrm{id})}\big]^2 = \frac{2\xi^2(gg^*\beta^2 - hh^*\alpha^2)^2}{[1 + (gg^* + hh^*)\xi]^2} + 2\alpha^2\beta^2 \tag{4.19}$$

The average distortion of the input qubit is given by

$$\overline{D_6} = \int_0^1 D_6(\alpha^2)\,\mathrm{d}\alpha^2 = \frac{1}{3} + \frac{2\xi^2[(gg^*)^2 + (hh^*)^2 - (gg^*)(hh^*)]}{3[1 + (gg^* + hh^*)\xi]^2} = \frac{1}{3} + \frac{2}{3}\left(\frac{(gg^*)^2 + (hh^*)^2 - (gg^*)(hh^*)}{[6 + (gg^* + hh^*)]^2}\right),$$

for the BH cloning machine $\xi = \frac{1}{6}$.

The fidelity of deletion is given by

$$F_6 = \langle\Sigma|\rho_\mathrm{y}|\Sigma\rangle = \frac{1 + \xi(gg^* - hh^*)M^2 + \xi(hh^*)}{1 + (gg^* + hh^*)\xi} = \frac{6 + (gg^* - hh^*)M^2 + (hh^*)}{6 + gg^* + hh^*},$$

for the BH cloning machine $\xi = \frac{1}{6}$.

In particular, For $a_0 = \frac{\sqrt{3}}{2}, a_1 = \frac{i}{2}, b_0 = \frac{i}{2}, b_1 = \frac{\sqrt{3}}{2}$, we get $gg^* = hh^* = 1$. In this case, we find that the fidelity of deletion and the average distortion is same as in the case of the BH cloning machine and the PB deleting machine.

General deletion machine

The general deletion machine can be prescribed as

$$U|0\rangle|0\rangle|Q\rangle \rightarrow |0\rangle|\Sigma\rangle|A_0\rangle + p_0|1\rangle|0\rangle|B_0\rangle + p_1|0\rangle|1\rangle|C_0\rangle \tag{4.20a}$$

$$U|0\rangle|1\rangle|Q\rangle \rightarrow (a_0|0\rangle|1\rangle + b_0|1\rangle|0\rangle)|Q\rangle \tag{4.20b}$$

$$U|1\rangle|0\rangle|Q\rangle \rightarrow (a_1|0\rangle|1\rangle + b_1|1\rangle|0\rangle)|Q\rangle \tag{4.20c}$$

$$U|1\rangle|1\rangle|Q\rangle \rightarrow |1\rangle|\Sigma\rangle|A_1\rangle + p_0|0\rangle|1\rangle|B_1\rangle + p_1|1\rangle|0\rangle|C_1\rangle \tag{4.20d}$$

where $|Q\rangle, |A_i\rangle, |B_i\rangle, |C_i\rangle$ $(i = 0, 1)$ and $|\Sigma\rangle$ have their usual meaning and a_i, b_i, p_i $(i = 0, 1)$ are complex numbers.

Due to the unitarity of the transformation (3.1) the following relations hold:

$$\left.\begin{array}{ll} |p_i|^2\langle B_i|B_i\rangle + |p_{1-i}|^2\langle C_i|C_i\rangle = 1 - \langle A_i|A_i\rangle & (i = 0, 1) \\ |a_i|^2 + |b_i|^2 = 1 \qquad (i = 0, 1) & \\ a_i a_{1-i}^* b_i b_{1-i}^* = 0 \qquad (i = 0, 1) & \\ p_i p_{1-i}^* \langle C_1|B_0\rangle p_i^* p_{1-i}\langle B_1|C_0\rangle = 0. & \end{array}\right\} \tag{4.21}$$

Further, we assume that

$$\langle A_i|Q\rangle = \langle B_i|Q\rangle = 0 = \langle C_i|Q\rangle = \langle A_0|A_i\rangle = 0. \tag{4.22}$$

The above constructed deleting machine is general in the sense that it reduces to the deleting machine discussed in this paper for the assigned values of a_i, b_i, p_i $(i = 0, 1)$. Moreover, it also gives a wide class of deleting machines.

Conclusion

In this work, we define a deleting machine which gives a slightly better result than the PB deleting machine. In addition, for some particular values of a_0, a_1, b_0, b_1, our deleting machine (3.1) acts like the PB deleting machine. Also here we observe that the concatenation of the WZ cloning machine and the PB deleting machine always gives the same result as obtained in the case of the WZ cloning machine and deleting machine (3.1). But the result obtained from the application of the BH cloning machine and deleting machine (3.1) on an unknown quantum state does not always coincide with the result obtained from the combination of the BH cloning machine and the PB deleting machine. The two results agree only when $a_0 = \frac{\sqrt{3}}{2}, a_1 = \frac{i}{2}, b_0 = \frac{i}{2}, b_1 = \frac{\sqrt{3}}{2}$. In this work, we mainly concentrate on the state-dependent WZ cloning machine and the state-independent BH cloning machine to clone an unknown quantum state but there also exist various types of state-dependent cloning machines which may give better results than the above two types.

Acknowledgment

The present work is supported by NBHM, project no 48/2/2000-R&D-II/1317 dated 15 November 2000, of Department of Atomic Energy, Government of India. The support is gratefully acknowledged.

References

[1] Wootters W K and Zurek W H 1982 *Nature* **299** 802
[2] Buzek V and Hillery M 1996 *Phys. Rev.* A **54** 1844
[3] Gisin N and Massar S 1997 *Phys. Rev. Lett.* **79** 2153
[4] Zurek W H 2000 *Nature* **404** 131
[5] Pati A K and Braunstein S L 2000 *Nature* **404** 164
[6] Pati A K and Braunstein S L 2000 *Preprint* quant-ph/0007121
[7] Kwek L C *et al* 2000 *Phys. Rev.* A **62** 052313
[8] Qiu D 2002 *Phys. Lett.* A **301** 112